

CONVOLUTIONS WITH KERNELS HAVING SINGULARITIES ON A SPHERE

BY

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Abstract. We prove that convolution with $(1 - |x|^2)^{-\alpha}$ and related convolutions are bounded from L^p to L^q for certain values of p and q . There is a unique choice of p which maximizes the measure of smoothing $1/p - 1/q$, in contrast with fractional integration where $1/p - 1/q$ is constant. We apply the results to obtain a priori estimates for solutions of the wave equation in which we sacrifice one derivative but gain more smoothing than in Sobolev's inequality.

1. Introduction. For convolutions on Euclidean n -space E_n with homogeneous kernels, the fractional integration theorem and the Calderón-Zygmund inequalities [3] give essentially best possible results. Such kernels have singularities at 0 and ∞ . In this paper we study the boundedness in various L^p norms of the convolution operator $Tf(x) = \int_{E_n} f(x-y)K(y) dy$ for K having its singularities on the unit sphere. The results are quite different in character from the case of homogeneous K . Such operators arise naturally in the study of the wave equation, and our results will give us new information about solutions of this equation.

More precisely, let Σ denote the unit sphere in E_n , let x, y denote points in E_n and x', y' denote points in Σ . Let dx' be Lebesgue measure on Σ normalized so that $\int_{E_n} f(x) dx = \int_0^\infty \int_\Sigma f(rx') dx' r^{n-1} dr$. For each nonnegative integer and $f \in C_{\text{com}}^\infty$ we define

$$(1) \quad T_{k+1}f(x) = \int_\Sigma \left(\frac{\partial}{\partial r} \right)^k f(x - ry')|_{r=1} dy'.$$

For $0 < \alpha < 1$ we define

$$(2) \quad T_\alpha f(x) = \int_{|y| \leq 1} f(x-y)(1 - |y|^2)^{-\alpha} dy.$$

For $\alpha \geq 1$ the integral (2) no longer makes sense as such because $(1 - |y|^2)^{-\alpha}$ has a nonintegrable singularity on Σ . However, we want to define T_α for nonintegral α in the range $0 < \alpha \leq (n+1)/2$ by interpreting (2) in a distribution sense. We do this

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by formally integrating by parts in the radial direction and neglect the boundary terms. Thus, if $k < \alpha < k+1$ we define

$$(3) \quad T_{\alpha} f = (-\tfrac{1}{2})^k \frac{\Gamma(\alpha-k)}{\Gamma(\alpha)} \int_{\Sigma} \int_0^1 (1-r^2)^{k-\alpha} \left(\frac{\partial}{\partial r} \frac{1}{r} \right)^k (r^{n-1} f(x-ry')) dr dy'.$$

We can now state our main results:

THEOREM 1. (a) $\|T_{\alpha} f\|_q \leq A_{\alpha} \|f\|_p$ provided $1 < p \leq 2 \leq q < \infty$ and $1/p - 1/q \leq (n+1-2\alpha)/2n$ for $0 < \alpha \leq (n+1)/2$.

(b) $\|T_{\alpha} f\|_{p'} \leq A_{\alpha} \|f\|_p$ for $p = (n+1)/(n+1-\alpha)$, $p' = (n+1)/\alpha$ and $0 < \alpha \leq (n+1)/2$.

(c) $\|T_{\alpha} f\|_q \leq A_{\alpha} \|f\|_p$ for $\frac{1}{2} \leq \alpha \leq (n+1)/2$ provided $(n+1)/(n+1-\alpha) \leq p \leq 2$ and $n/q = \alpha - 1/p$, or $n/(n+\frac{1}{2}-\alpha) \leq p \leq (n+1)/(n+1-\alpha)$ and $1/q = \alpha - n/p'$.

We remark that the amount of smoothing in (b) is $1/p - 1/p' = (n+1-2\alpha)/(n+1)$ which is more than in (a) provided $n \geq 2$. On the other hand (a) is applicable to a wider range of p 's. It is also possible to interpolate between these results in case $\alpha < \frac{1}{2}$. We leave the details to the interested reader.

In §2 we prove Theorem 1. In §3 we give applications to the wave equation. In §4 we prove a similar result for spherical convolutions.

2. Proof of Theorem 1. Let

$$\phi_{\alpha}(y) = \begin{cases} (1-|y|^2)^{-\alpha} & \text{if } |y| < 1, \\ 0 & \text{if } |y| \geq 1. \end{cases}$$

If $\alpha < 1$, $\phi_{\alpha} \in L^1$ so we can compute its Fourier transform. The result

$$(4) \quad \hat{\phi}_{\alpha}(\xi) = (2\pi)^{-n/2} 2^{-\alpha} \Gamma(1-\alpha) |\xi|^{\alpha-n/2} J_{n/2-\alpha}(|\xi|)$$

is well known [1] or [8]. We thus have

$$(5) \quad T_{\alpha}(f)^{\wedge}(\xi) = (2\pi)^{-n/2} 2^{-\alpha} \Gamma(1-\alpha) |\xi|^{\alpha-n/2} J_{n/2-\alpha}(|\xi|) \hat{f}(\xi)$$

for $0 \leq \alpha < 1$. But in fact (5) holds for all $\alpha \neq 1, 2, \dots$ (the poles of $\Gamma(1-\alpha)$). For the right-hand side of (5) is a single-valued analytic function of α in the complex plane minus the positive integers. On the other hand, the Fourier transform of the right-hand side of (3), for fixed k , is also a single-valued analytic function of α in $\text{Re } (\alpha) < k+1$ minus $\alpha = k, k-1, \dots, 1$. These two functions agree for real α in the interval $0 < \alpha < 1$, hence by the principle of analytic continuation they are equal for $\alpha \neq 1, 2, \dots$

We can also compute $T_k f^{\wedge}(\xi)$ for $k = 1, 2, \dots$. In fact $T_k f$ is the convolution of f with the distribution $(\partial/\partial r)^k \sigma$, where σ is Lebesgue measure on Σ , regarded as a finite measure in E_n , and $\partial/\partial r$ is radial differentiation. Now we have the well-known formula [1]:

$$(6) \quad \hat{\sigma}(\xi) = (2\pi)^{n/2} |\xi|^{1-n/2} J_{n/2-1}(|\xi|).$$

Since $\partial/\partial r = (1/|x|)x \cdot \nabla$ and $|x| = 1$ on the support of σ , we have

$$\left(\left(\frac{\partial}{\partial r} \right)^k \sigma \right)^\wedge(\xi) = (2\pi)^{n/2} \left(\sum_{j=1}^n \frac{\partial}{\partial \xi_j} \xi_j \right)^k (|\xi|^{1-n/2} J_{n/2-1}(|\xi|)).$$

Using the recursion relations [9]

$$(7) \quad (d/dt)(t^{-s} J_s(t)) = -t^{-s} J_{s+1}(t)$$

and

$$(8) \quad J_{s+1}(t) = (2s/t)J_s(t) - J_{s-1}(t)$$

and the fact that

$$\sum_{j=1}^n \frac{\partial}{\partial \xi_j} \xi_j = nId + |\xi| \frac{\partial}{\partial |\xi|},$$

we easily obtain

$$(9) \quad \left(\left(\frac{\partial}{\partial r} \right)^k \sigma \right)^\wedge(\xi) = \sum_{i=1}^{k+1} c_i |\xi|^{i-n/2} J_{n/2-i}(|\xi|)$$

for certain constants c_i depending on n and k . Thus we have

$$(10) \quad T_k f^\wedge(\xi) = \sum_{i=1}^k c_i |\xi|^{i-n/2} J_{n/2-i}(|\xi|) \hat{f}(\xi).$$

We now define operators S_α , for α in the strip $0 \leq \operatorname{Re}(\alpha) \leq (n+1)/2$, and $f \in C_{\text{com}}^\infty$, by

$$(11) \quad S_\alpha f^\wedge(\xi) = |\xi|^{\alpha-n/2} J_{n/2-\alpha}(|\xi|) \hat{f}(\xi).$$

It clearly suffices to prove estimates (a) (b) and (c) for S_α in place of T_α .

We need the classical estimates on the size of the Bessel function [8, (11.10) and (11.11)],

$$(12) \quad |t^{-(a+ib)} J_{a+ib}(t)| \leq c_a e^{c|b|} (1+t)^{-a-1/2}$$

for $0 < t < \infty$. This is usually proved for $a > -\frac{1}{2}$ but it actually holds for all a by (8). This estimate is sufficient to establish part (a) of the theorem as a consequence of the theorem of Hardy and Littlewood [3] that asserts that the operator $\mathcal{F}^{-1}(m(\xi)\hat{f}(\xi))$ is bounded from L^p to L^q provided $1 < p \leq 2 \leq q < \infty$ and

$$(13) \quad |m(\xi)| \leq c|\xi|^{-t} \quad \text{for } 1/p - 1/q = t/n.$$

For S_α has this form with $m(\xi) = |\xi|^{\alpha-n/2} J_{n/2-\alpha}(|\xi|)$.

Thus (12) implies (13) for any $t \leq n/2 - \alpha + \frac{1}{2}$ which proves (a).

To prove (b) we use the fact that S_α is an analytic function of α and the interpolation theorem of Stein [7] for such analytic families of operators. We note that (12) restricts the growth of $\|S_\alpha f\|_2$ in the strip $0 \leq \operatorname{Re}(\alpha) \leq (n+1)/2$ so that the theorem is applicable. On the boundary $\operatorname{Re}(\alpha) = (n+1)/2$ we have $\|S_\alpha f\|_2 \leq$

$c \exp(c \operatorname{Im}(\alpha)) \|f\|_2$ by (12) and the Plancherel theorem. On the boundary $\operatorname{Re}(\alpha)=0$ we use (5) and the obvious estimate $\|T_\alpha f\|_\infty \leq \|f\|_1$ since $|\phi_\alpha(x)| \leq 1$. Since $|\Gamma(1+ib)| = (\pi b / \sinh b)^{1/2}$ we obtain $\|S_\alpha f\|_\infty \leq c \exp(c \operatorname{Im}(\alpha)) \|f\|_1$ for $\operatorname{Re}(\alpha)=0$. Stein's theorem now implies part (b). We obtain (c) from (a) and (b) immediately by the M. Riesz interpolation theorem, if $\alpha > \frac{1}{2}$. For $\alpha = \frac{1}{2}$, we see that $T_{1/2}$ maps L^1 to weak L^2 since $\phi_{1/2}$ is in weak L^2 . Applying the Marcinkiewicz interpolation theorem [10] and a duality argument [3] we obtain (c).

REMARKS. For $0 \leq \alpha \leq 1$ we can show that estimate (c) is sharp, in the sense that if $\|T_\alpha f\|_q \leq M \|f\|_p$ then $1/q \geq (\alpha - 1 + 1/p)/n$. For if we choose f to be ϕ_β it is not difficult to show that $|T_\alpha \phi_\beta(x)| \geq c|x|^{-\beta-\alpha+1}$ for $|x| \leq \frac{1}{2}$.

It should be possible to prove similar results for kernels obtained by replacing $|x|^2$ by any nondegenerate quadratic form by using the computations of [1, Chapter III, §2].

3. The wave equation. The Cauchy problem for the wave equation in n -space variables x and time variable t

$$(14) \quad \partial^2 u(x, t) / \partial t^2 = \Delta_x u(x, t),$$

$$(15) \quad u(x, 0) = f(x), \quad \partial u(x, 0) / \partial t = g(x)$$

for $f, g \in C_{\text{com}}^\infty(E_n)$ has a unique solution

$$(16) \quad u(x, t) = \mathcal{F}^{-1} \left(\hat{f}(\xi) \cos t|\xi| + \hat{g}(\xi) \frac{\sin t|\xi|}{|\xi|} \right).$$

Now we have

$$J_{1/2}(t) = \left(\frac{2}{\pi} \right)^{1/2} \frac{\sin t}{t^{1/2}} \quad \text{and} \quad J_{-1/2}(t) = \left(\frac{2}{\pi} \right)^{1/2} \frac{\cos t}{t^{1/2}}$$

so that (16) becomes

$$(17) \quad u(x, t) = \left(\frac{\pi}{2} \right)^{1/2} \delta(t) S_{(n+1)/2} \delta(t^{-1}) f + \left(\frac{\pi}{2} \right)^{1/2} t \delta(t) S_{(n-1)/2} \delta(t^{-1}) g$$

where $\delta(t)$ is the dilation operator

$$(18) \quad \delta(t)f(x) = f(tx).$$

We define the p -energy of u by

$$(19) \quad E_p(t) = \int_{E_n} \left\{ \left| \frac{\partial u}{\partial t}(x, t) \right|^p + |\nabla_x u(x, t)|^p \right\} dx.$$

Thus $E_p(0) = \|g\|_p^p + \|\nabla f\|_p^p$. The classical conservation of energy principle states that $E_2(t)$ is a constant. Littman [5] has shown that there is no analogue of this statement for $p \neq 2$; in fact there exist weak solutions with $E_p(0) < \infty$ and $E_p(t) = \infty$ for all $t \neq 0$.

Here we obtain estimates on the L^q norm of $u(x, t)$ for fixed $t \neq 0$ in terms of the p -energy at time $t=0$ for certain values of p and q .

THEOREM 2. Let $u(x, t)$ be given by (16), $n \geq 2$.

(a) For any $t \neq 0$

$$\left(\int_{E_n} |u(x, t)|^q dx \right)^{1/q} \leq c(t) E_p(0)^{1/p},$$

where

$$\frac{n}{q} = \frac{n-1}{2} - \frac{1}{p'}, \quad \text{for } 2 \frac{n+1}{n+3} \leq p \leq 2$$

and

$$\frac{1}{q} = \frac{n}{p} - \frac{n+1}{2} \quad \text{for } 2 \frac{n}{n+2} \leq p \leq 2 \frac{n+1}{n+3}$$

($p > 1$ in case $n=2$).

(b) For any $t \neq 0$ and any integer $k \geq 1$

$$\|u(\cdot, t): L_{k-1}^q\| \leq c(t)(\|g: L_{k-1}^p\| + \|f: L_k^p\|)$$

for p and q related as above. Here the norm

$$\|f: L_k^p\| = \left(\sum_{|\alpha| \leq k} \left\| \left(\frac{\partial}{\partial x} \right)^\alpha f \right\|_p^p \right)^{1/p}.$$

REMARKS. The inequalities remain valid for any f and g for which the right side is finite. This follows by the usual limit argument.

The Sobolev inequalities imply

$$\|f: L_{k-1}^r\| \leq c \|f: L_k^p\| \quad \text{for } 1/r = 1/p - 1/n, \quad 1 < p, r < \infty.$$

Since $u(x, 0) = f(x)$ we may compare this with our results for $t \neq 0$. We find that for the appropriate choice of p (in fact $2n/(n+2) < p < 2$) we have $q > r$, i.e. $u(x, t)$ is better behaved at any time $t \neq 0$ than at time $t=0$.

We can use this remark to illustrate the phenomenon of focusing of singularities. Let $p = 2(n+1)/(n+3)$ so that $q = 2(n+1)/(n-1)$. We then choose k so that $1/q - (k-1)/n < 0$ but $1/p - k/n > 0$. We can then find an $f \in L_k^p$ which is unbounded in any open set. Forming u from f with say $g=0$, we find that $u(\cdot, t)$ for any $t \neq 0$ is continuous and bounded by (b) and the Sobolev embedding theorem. Similarly we can construct weak solutions $u(x, t)$ which are continuously differentiable in x for all $t \neq 0$ but not differentiable for $t=0$.

Proof of Theorem 2. We note that (b) follows from (a) since any space derivative $(\partial/\partial x)^\alpha u(x, t)$ is again a solution of the wave equation with Cauchy data $(\partial/\partial x)^\alpha f$ and $(\partial/\partial x)^\alpha g$.

To prove (a) we observe from (17) that it suffices to prove

$$(20) \quad \|S_{(n-1)/2} g\|_q \leq c \|g\|_p$$

and

$$(21) \quad \|S_{(n+1)/2} f\|_q \leq c \|\nabla f\|_p.$$

Now (20) is just a special case of Theorem 1, and (21) is equivalent to

$$(22) \quad \left\| \mathcal{F}^{-1} \left(\frac{\cos |\xi|}{|\xi|} \hat{f}(\xi) \right) \right\|_q \leq c \|f\|_p.$$

Let ψ be a function in $C^\infty(E^n)$ which vanishes in a neighborhood of the origin and is identically one for large ξ . Then the Hardy-Littlewood multiplier theorem implies

$$(23) \quad \left\| \mathcal{F}^{-1}(1 - \psi(\xi)) \frac{\cos |\xi|}{|\xi|} \hat{f}(\xi) \right\|_q \leq c \|f\|_p$$

for the values of p and q given above since $1 < p \leq 2 \leq q < \infty$ and $1/p - 1/q \geq 1/n$.

Thus we must study the operator $\mathcal{F}^{-1}(\psi(\xi)(\cos |\xi|/|\xi|)\hat{f}(\xi))$. We introduce related analytic families defined by

$$(24) \quad U_\alpha^{(j)}(f) = \mathcal{F}^{-1}(\psi(\xi)\xi_j|\xi|^{\alpha-n/2-1}J_{n/2-\alpha-1}(|\xi|)\hat{f}(\xi)), \quad j = 1, \dots, n,$$

and imitate the proof of Theorem 1. Part (a) of Theorem 1 for $U_\alpha^{(j)}$ follows by exactly the same reasoning. To prove the analogue of (b) we must show for $\text{Re } \alpha = 0$ that

$$\mathcal{F}^{-1}(|\xi|^{\alpha-n/2-1}\xi_j\psi(\xi)J_{n/2-\alpha-1}(|\xi|))$$

is in L^∞ and

$$(25) \quad \left\| \mathcal{F}^{-1}(|\xi|^{ib-n/2-1}\psi(\xi)\xi_jJ_{n/2-1+ib}(|\xi|)) \right\|_\infty \leq ce^{c|b|}.$$

To prove this we begin with formula (4) for $\alpha = ib$ and deduce, using (7) and (8), that

$$\begin{aligned} (x_j\phi_{ib})^\wedge(\xi) &= c(b)(\partial/\partial\xi_j)(|\xi|^{-n/2-ib}J_{n/2+ib}(|\xi|)) \\ &= c(b)(\xi_j/|\xi|)(-|\xi|^{-n/2-ib}J_{n/2+1+ib}(|\xi|)) \\ &= c(b)\xi_j|\xi|^{-n/2-1-ib}J_{n/2-1+ib}(|\xi|) \\ &\quad - \xi_j|\xi|^{-n/2-2-ib}J_{n/2+ib}(|\xi|)c(b)(n+2ib) \end{aligned}$$

hence

$$\begin{aligned} (26) \quad \mathcal{F}^{-1}(\psi(\xi)\xi_j|\xi|^{-n/2-1-ib}J_{n/2-1+ib}(|\xi|)) &= \frac{1}{c(b)} \mathcal{F}^{-1}(\psi(\xi)(x_j\phi_{ib})^\wedge) \\ &\quad + \frac{1}{(n+2ib)c(b)} \mathcal{F}^{-1}(\psi(\xi)\xi_j|\xi|^{-2}\hat{\phi}_{ib}). \end{aligned}$$

Now $\psi(\xi)$ is the Fourier-Stieltjes transform of a bounded measure so the first term on the right is in L^∞ . The second term is in L^∞ because the multiplier operator $\mathcal{F}^{-1}(\xi_j|\xi|^{-1}\hat{f}(\xi))$ (the so-called j th Riesz transform) is bounded on any L^p , $1 < p < \infty$, the operator $\mathcal{F}^{-1}(\psi(\xi)|\xi|^{-1}\hat{f}(\xi))$ sends L^p to L^∞ for any $p > n$, and $\phi_{ib} \in L^1 \cap L^\infty$. An examination of the various constants shows that the growth in b is at most exponential.

Thus the same estimates hold for $U_\alpha^{(j)}$ as for S_α . In particular setting $\alpha = (n-1)/2$ we obtain

$$(27) \quad \left\| \mathcal{F}^{-1}(\psi(\xi)\xi_j|\xi|^{-2}\cos|\xi|\hat{f}(\xi)) \right\|_q \leq c \|f\|_p.$$

Applying the j th Riesz transform and summing over j we obtain

$$(28) \quad \left\| \mathcal{F}^{-1} \left(\psi(\xi) \frac{\cos |\xi|}{|\xi|} \hat{f}(\xi) \right) \right\|_q \leq c \|f\|_p$$

which, together with (23) gives the desired result.

Other applications of these ideas to the wave equation will be given elsewhere.

4. Spherical convolutions. In analogy with (2) we define for $n \geq 2$

$$(29) \quad \tilde{T}_\alpha f(x') = \int_{\Sigma} f(y') |x' \cdot y'|^{-\alpha} dy'$$

for $0 < \alpha < 1$ and f defined on Σ . Later we will discuss the extension to $\alpha \geq 1$.

THEOREM 3. (a) $\|\tilde{T}_\alpha f\|_q \leq A_\alpha \|f\|_p$ provided $1 < p \leq 2 \leq q < \infty$ and $1/p - 1/q \leq (n-2\alpha)/(2(n-1))$.

(b) $\|\tilde{T}_\alpha f\|_{p'} \leq A_\alpha \|f\|_p$ for $p = n/(n-\alpha)$, $p' = n/\alpha$.

(c) $\|\tilde{T}_\alpha f\|_q \leq A_\alpha \|f\|_p$ for $\alpha \geq \frac{1}{2}$ provided $n/(n-\alpha) \leq p \leq 2$ and $(n-1)/q = \alpha - 1/p'$, or $(n-1)/(n-\frac{1}{2}-\alpha) \leq p \leq n/(n-\alpha)$ and $1/q = \alpha - (n-1)/p'$.

As one would expect, this is the direct analogue of Theorem 1 for E^{n-1} , since $\tilde{T}_\alpha$ is a spherical convolution with a kernel having a singularity on the $n-2$ dimensional equator of the $n-1$ sphere. However, there seems to be no simple way to deduce Theorem 3 from Theorem 1.

We shall give a proof which is quite different from the proof of Theorem 1. We realize $\tilde{T}_\alpha$ as the real part of an operator $\tilde{S}_\alpha$ derived from the Fourier transform, and prove (a) and (b) for $\tilde{S}_\alpha$. We obtain (c) as before by interpolation.

We begin by deriving some properties of the Fourier transform.

Denote by $L^{p,\infty}(E^n)$ the space of functions $f(x)$ satisfying $m\{x : |f(x)| > s\} \leq M^p/s^p$ for all s , $0 < s < \infty$, and let $\|f\|_{p,\infty}^*$ be the least such M . It follows from Hunt's theorem [4] that the Fourier transform is a bounded operator from $L^{p,\infty}$ to $L^{p',\infty}$ for $1 < p < 2$ and $1/p + 1/p' = 1$.

LEMMA 1. Let $f(x) = \Omega(x')|x|^{-n/p}$, $1 < p < \infty$. Then $f \in L^{p,\infty}$ if and only if $\Omega \in L^p(S^{n-1})$.

Proof. Note $|f(x)| > s$ if and only if $|x| < (|\Omega(x')|/s)^{p/n}$ so

$$\begin{aligned} m\{x : |f(x)| > s\} &= \int_{S^{n-1}} \int_0^{(|\Omega(x')|/s)^{p/n}} r^{n-1} dr dx' \\ &= \frac{1}{n} \int_{S^{n-1}} \left(\frac{|\Omega(x')|}{s} \right)^p dx'. \end{aligned}$$

LEMMA 2. Let $1 < p < 2$, $\Omega \in L^p(S^{n-1})$ and $f(x) = \Omega(x')|x|^{-n/p}$. Then the Fourier transform $\hat{f}(\xi) = \psi(\xi')|\xi|^{-n/p'}$ for some $\psi \in L^{p'}(S^{n-1})$, $1/p + 1/p' = 1$.

Proof. By Lemma 1 $f \in L^{p, \infty}$ hence $\hat{f} \in L^{p', \infty}$ by Hunt's theorem. Now

$$\begin{aligned}\hat{f}(\xi) &= \int \exp(ix \cdot \xi) f(x) dx = \int \exp(ix' \cdot \xi' |x| |\xi|) \Omega(x') |x|^{-n/p} dx \\ &= |\xi|^{-n/p'} \int \exp(ix \cdot \xi') \Omega(x') |x|^{-n/p} dx = |\xi|^{n/p'} \hat{f}(\xi')\end{aligned}$$

where $\xi' = \xi/|\xi|$. Thus f has the desired form. Applying Lemma 1 again shows $\psi \in L^{p'}$.

DEFINITION. Let $\tilde{S}_p$, $1 < p < 2$, denote the map $\Omega \rightarrow \psi$ as in Lemma 2; $S_p \Omega(\xi') = \mathcal{F}(\Omega(x') |x|^{-n/p})(\xi')$.

As an immediate corollary of Lemma 2, we have

$$(30) \quad \|\tilde{S}_p \Omega\|_{p'} = A_p \|\Omega\|_p \quad \text{for all } \Omega \in L^p, \quad 1 < p < 2.$$

LEMMA 3. Let $1 < p < 2$. Then

$$\tilde{S}_p \Omega(\xi') = \lim_{k \rightarrow \infty} \int_{|x| \leq N_k} \Omega(x') |x|^{-n/p} \exp(ix \cdot \xi') dx$$

in the L^2 norm, for any sequence $N_k \rightarrow \infty$ sufficiently rapidly.

Proof. Since $f(x) = \Omega(x') |x|^{-n/p} \in L^{p, \infty}$ and $L^{p, \infty} \subset L^1 + L^2$ we have

$$\int_{|x| \leq N} \Omega(x') |x|^{-n/p} \exp(ix \cdot \xi) dx$$

converging to $\hat{f}(\xi)$ in the L^2 norm on the spherical shell $\frac{1}{2} \leq |\xi| \leq 2$. It follows that for subsequences $N'_k \rightarrow \infty$ fast enough we must have convergence in L^2 on some sphere $|\xi| = a$ for $\frac{1}{2} \leq a \leq 2$. By homogeneity we obtain convergence on the sphere $|\xi| = 1$ for the subsequences aN'_k .

LEMMA 4. Let $1 < p < n/(n-1)$ so that $0 < n/p' < 1$. Let $\alpha = n/p'$. Then

$$\tilde{S}_p \Omega(\xi') = \int_{S^{n-1}} \Omega(x') (a_p + ib_p \operatorname{sgn}(x' \cdot \xi')) |x' \cdot \xi'|^{-\alpha} dx'$$

where

$$a_p = \lim_{N \rightarrow \infty} \int_0^N r^{\alpha-1} \cos r dr \quad \text{and} \quad b_p = \lim_{N \rightarrow \infty} \int_0^N r^{\alpha-1} \sin r dr,$$

for all $\Omega \in L^p$.

Proof. It follows by integration by parts not only that a_p and b_p exist and are finite, but also

$$\sup_N \left| \int_0^N r^{\alpha-1} \cos r dr \right| = A_p < \infty \quad \text{and} \quad \sup_N \left| \int_0^N r^{\alpha-1} \sin r dr \right| = B_p < \infty$$

for $0 < \alpha < 1$.

We compute

$$\int_{|x| \leq N} \Omega(x') |x|^{-n/p} \exp(ix \cdot \xi') dx = \int_{S^{n-1}} \Omega(x') \left[\int_0^N \exp(irx' \cdot \xi') r^{\alpha-1} dr \right] dx'.$$

Now

$$\begin{aligned} \int_0^N \exp(irx' \cdot \xi') r^{\alpha-1} dr &= \int_0^N (\cos rx' \cdot \xi' + i \sin rx' \cdot \xi') r^{\alpha-1} dr \\ &= |x' \cdot \xi'|^{-\alpha} \int_0^{N|x' \cdot \xi'|} (r^{\alpha-1} \cos r + i \operatorname{sgn}(x' \cdot \xi') r^{\alpha-1} \sin r) dr \end{aligned}$$

which converges to $|x' \cdot \xi'|^{-\alpha}(a_p + ib_p \operatorname{sgn}(x' \cdot \xi'))$ if $x' \cdot \xi' \neq 0$. Now for almost every $\xi' \in S^{n-1}$ we have $|x' \cdot \xi'|^{-\alpha} |\Omega(x')| \in L^1(S^{n-1})$ so that the integrand

$$\left| \Omega(x') \int_0^N \exp(irx' \cdot \xi') r^{\alpha-1} dr \right| \leq (A_p + B_p) |\Omega(x')| |x' \cdot \xi'|^{-\alpha} \in L^1(S^{n-1}).$$

Thus we may apply the dominated convergence theorem to conclude

$$\begin{aligned} \lim_{N \rightarrow \infty} \int_{|x| \leq N} \Omega(x') |x|^{-n/p} \exp(ix \cdot \xi') dx \\ = \int_{S^{n-1}} \Omega(x') (a_p + ib_p \operatorname{sgn}(x' \cdot \xi')) |x' \cdot \xi'|^{-\alpha} dx' \end{aligned}$$

for almost every $y' \in S^{n-1}$.

Lemma 4 shows that $\tilde{T}_\alpha \Omega = \operatorname{Re} \tilde{S}_p \Omega$ for real valued Ω and $\alpha = n/p'$, $0 < \alpha < 1$. Thus (30) established part (b) of Theorem 3. To establish part (a) we expand Ω in spherical harmonics and compute S_p in terms of the expansion.

Let $Y_k(x)$ be any harmonic polynomial homogeneous of degree k . We use the formula [2], [11]

$$(31) \quad \mathcal{F}(|x|^{\alpha-n-k} Y_k(x)) = \gamma_{k,\alpha} Y_k(\xi) |\xi|^{-k-\alpha},$$

where

$$(32) \quad \gamma_{k,\alpha} = \pi^{-\alpha+n/2} i^k \Gamma(\alpha/2+k/2) / \Gamma(k/2+n/2-\alpha/2).$$

Thus if $\Omega = \sum_{k=0}^N Y_k(x')$ we have

$$(33) \quad \tilde{S}_p \Omega = \sum_{k=0}^N \gamma_{k,\alpha} Y_k(x'), \quad \text{where } \alpha = \frac{n}{p'},$$

and this uniquely determines $\tilde{S}_p$. We can study the properties of such a spherical harmonic multiplier transform using a result of Marcinkiewicz and Zygmund [6].

Let Ω be any sufficiently regular function on Σ . Then Ω has a spherical harmonic expansion $\Omega = \sum_{k=0}^{\infty} a_k Y_k(x')$ where the Y_k are spherical harmonics of degree k normalized so that $\int_{\Sigma} |Y_k(x')|^2 dx = 1$. For different choices of Ω we must choose different Y_k , but this will not matter. The system $\{Y_k\}$ is orthonormal and satisfies

$$(34) \quad \|Y_k\|_{\infty} \leq (\dim H_k)^{1/2} \leq M k^{(n-2)/2},$$

where H_k is the space of all spherical harmonics of degree k . Thus by Theorems 1, 2, 3, 4 and the final remark of [6],

$$(35) \quad \|\Omega\|_q \leq A \left(\sum_{k=0}^{\infty} |a_k|^r (1+k)^{(n/2-(n-1)/q)r-1} \right)^{1/r} \quad \text{for } 2 \leq q < \infty, q' \leq r \leq q,$$

and

$$(36) \quad \left(\sum_{k=0}^{\infty} |a_k|^r (1+k)^{(n/2-(n-1)/p)r-1} \right)^{1/r} \leq A \|\Omega\|_p \quad \text{for } 1 < p \leq 2, p \leq r \leq p'.$$

LEMMA 5. *Let c_k be any sequence of complex numbers satisfying $|c_k| \leq B(k+1)^{-\beta}$. If $S\Omega(x') = \sum_{k=0}^{\infty} a_k c_k Y_k(x')$ then $\|S\Omega\|_q \leq A^2 B \|\Omega\|_p$ provided $1 < p \leq 2 \leq q < \infty$ and $1/p - 1/q = \beta/(n-1)$.*

Proof. We imitate the proof of Hardy-Littlewood for Fourier series. Using (35) and (36) with $r=2$, we have

$$\begin{aligned} \|S\Omega\|_q &\leq AB \left(\sum_{k=0}^{\infty} |a_k|^2 (1+k)^{-2\beta+n-2(n-1)/q-1} \right)^{1/2} \\ &= AB \left(\sum_{k=0}^{\infty} |a_k|^2 (1+k)^{n-2(n-1)/p-1} \right)^{1/2} \leq A^2 B \|\Omega\|_p. \end{aligned}$$

It is now a simple matter to deduce part (a) of Theorem 3 for $\tilde{S}_{n/(n-\alpha)}$ in place of $\tilde{T}_\alpha$ for $0 < \alpha < n/2$. For by (32) and Stirling's formula

$$(37) \quad |\gamma_{k,\alpha}| \leq c_\alpha (1+k)^{\alpha-n/2}$$

so Lemma 5 applies with $\beta = n/2 - \alpha$.

This completes the proof of Theorem 3. It is possible to extend definition (29) to nonintegral $\alpha > 1$ by formally integrating by parts along great circles through x' . It is also possible to extend Lemma 4 to the range $0 < \alpha < n/2$, α nonintegral, with some slight modifications when $n=3$ due to boundary terms. Theorem 3 will continue to hold.

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